Journal of Engineering Mathematics, Vol. 3, No. 2, April 1969
Wolters-Noordhoff Publishing – Groningen
Printed in the Netherlands

Variational Principles for Steady Heat Conduction With Mixed Boundary Conditions*

R. S. BRAND

(University of Connecticut, Storrs, Connecticut, U.S.A.)

F. J. LAHEY

(United Aircraft Research Laboratories, East Hartford, Connecticut, U.S.A.)

(Received June 6, 1968)

SUMMARY

For the boundary value problem of steady heat conduction with general boundary conditions a variational problem is formulated by adding a simple surface integral to Butler's volume integral.

1. Introduction

There has been considerable interest recently in formulating the problem of heat conduction as a variational problem. For example, Hays [1] gives an integral which takes on a stationary value when the temperature distribution satisfies the heat conduction equation in a region R, provided that at all points of the surface of R either the temperature is prescribed or the normal heat flux vanishes. Hays' formulation is applicable to both time-dependent and steady problems, and the conductivity and thermal capacity may be any given functions of the temperature. Butler [2] proposes a much simpler integral for steady problems with the same type of boundary conditions.

However, one often has the normal heat flux prescribed, rather than vanishing, on a portion of the bounding surface, or the even more complicated case when neither the temperature nor the normal heat flux is given, but rather a relation between them exists, as, for example, a surface heated or cooled by convection or radiation. It is the purpose of this note to show that these more complicated problems may be expressed as variational problems by adding a simple surface integral to Butler's volume integral.

Biot [3] has also given variational principles for heat flow problems and has included the types of boundary conditions considered here. However, he is primarily interested in time-dependent problems, and his integrals do not seem to reduce in the steady case to the simple forms given in the present paper.

2. Problem I: Normal Heat Flux Prescribed

Let it be required to find the steady temperature distribution, $T(x_1, x_2, x_3)$, in a region R, bounded by a surface S, subject to the boundary conditions

T is prescribed on S_1, a portion of S. (1)

$q_i n_i$ is prescribed on S_2, the remainder of S. (2)

Here, q_i denotes the Cartesian components of the heat flux vector and n_i denotes the components of the outer unit normal to S. The heat flux is related to the temperature field by the Fourier law of heat conduction,

$$q_i = -KT_{,i}\,, \tag{3}$$

* This research was supported by the National Science Foundation, Grant GK-108.

where $K=K(T)$ is any given function. The conservation of energy requires that

$$q_{i,i}=0 \text{ in } R\,. \tag{4}$$

Let $T^*(x_1, x_2, x_3)$ be the temperature distribution which satisfies the system (1), (2), (3), (4); and let K^* and q_i^* be the corresponding conductivity and heat flux. Further, let $T(x_1, x_2, x_3)$ be a neighboring function to T^*, satisfying the same boundary conditions. Then T may be expressed as

$$T=T^*+\varepsilon\eta\,, \tag{5}$$

where η is an arbitrary function of the co-ordinates and ε is a small parameter. The conductivity corresponding to the temperature field T is

$$K=K^*+\varepsilon\eta K'(T^*)\,, \tag{6}$$

where $K'(T^*)$ denotes the derivative of $K(T)$ with respect to T, evaluated at $T=T^*$. Similarly,

$$q_i=-[K^*+\varepsilon\eta K'(T^*)](T^*_{,i}+\varepsilon\eta_{,i})\,,$$

which is, to the first order in ε,

$$q_i=q_i^*-\varepsilon(\eta K^*)_{,i}\,. \tag{7}$$

Since T and T^* must agree on S_1, and $q_i n_i$ and $q_i^* n_i$ agree on S_2,

$$\eta=0 \text{ on } S_1\,; \tag{8}$$

$$(\eta K^*)_{,i}n_i=0 \text{ on } S_2\,. \tag{9}$$

Define $H(T)$ to be

$$H(T)=\int K(T)dT\,. \tag{10}$$

Then, for T near to T^*

$$H(T)=H^*+\varepsilon\eta K^*\,. \tag{11}$$

It can now be shown that the following integral assumes a stationary value when $T=T^*$:

$$I=\int_R q_i q_i dv+\int_{S_2} q_i n_i H\, ds\,. \tag{12}$$

If one expresses the right side of (12) in terms of starred functions and variations, one can then find

$$(dI/d\varepsilon)_{\varepsilon=0}=\int_R -q_i^*(\eta K^*)_{,i}dv+\int_{S_2} n_i[-H^*(\eta K^*)_{,i}+q_i^*\eta K^*]dS\,. \tag{13}$$

The volume integral in (13) may be written

$$-\int_R q_i^*(\eta K^*)_{,i}dv=-\int_R (q_i^*\eta K^*)_{,i}dv+\int_R \eta K^* q_{i,i}^* dv\,. \tag{14}$$

The first integral on the right side of (14) may be converted to a surface integral, and the second vanishes because q_i^* must satisfy equation (4). Also, in (13), the first term in the surface integral vanishes by the boundary condition (9). Hence, (13) becomes

$$(dI/d\varepsilon)_{\varepsilon=0}=-\int_S q_i^*\eta K^* n_i ds+\int_{S_2} q_i^*\eta K^* n_i ds\,. \tag{15}$$

But in (15), there is no contribution from the integration over S_1, because η vanishes there by boundary condition (8). Hence,

$$(dI/d\varepsilon)_{\varepsilon=0}=0 \tag{16}$$

which proves that I is stationary when $T=T^*$.

3. Problem II: Normal Heat Flux a Function of Temperature

This is the same as Problem I, except that boundary condition (2) is replaced by

$$q_i n_i = F(T) \text{ on } S_2 , \qquad (17)$$

with F a specified function but neither q_i nor T given on S_2.

Define $G(T)$ to be the function

$$G(T) = \int K(T)F(T)dT . \qquad (18)$$

If, as before, T^* is the temperature distribution which solves the problem, and $\varepsilon\eta$ is its variation,

$$G(T) = G^* + \varepsilon\eta K^* F^* . \qquad (19)$$

A functional which is stationary in this case is

$$J = \int_R q_i q_i dv + \int_{S_2} G ds . \qquad (20)$$

Following essentially the same procedure as in Problem I, one finds

$$(dJ/d\varepsilon)_{\varepsilon=0} = -\int_S q_i^* \eta K^* n_i ds + \int_{S_2} \eta K^* F^* ds . \qquad (21)$$

Since η vanishes on S_1, this can be written

$$(dJ/d\varepsilon)_{\varepsilon=0} = \int_{S_2} \eta K^* (F^* - q_i^* n_i) ds , \qquad (22)$$

which is seen to vanish because of (17).

REFERENCES

[1] D. F. Hays; Variational Formulation of the Heat Equation, Chapter 2 in *Non-equilibrium Thermodynamics, Variational Techniques, and Stability*. R. J. Donnelly et al., eds. U. of Chicago Press, 1966.

[2] H. W. Butler; A minimum principle for heat conduction. *Proc. 5th U.S. National Congress of Appl. Mech. A.S.M.E.*, New York, p. 739, 1966.

[3] M. A. Biot; New Methods in Heat Flow Analysis, With Applications to Flight Structures. *Jour. Aero. Sci.*, 24, pp. 857–873, 1957.